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PROBLEMS OF CONFLICT CONTROL OF HIGH DIMENSIONALITY FUNCTIONAL SYSTEMS†

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A minimax control problem with a performance index which is the sum of two terms is considered for a system with a delay. The first of these two terms in the Euclidean norm of the set of deviations of the motion of the system at specified instants of time from the stipulated objectives, while the second term is an integral-quadratic penalty which is imposed on the form of the control actions. The problem arises in a differential game. In this case, the history of the motion serves as the information for the strategies. A functional treatment of the control process in question is given which is based on an original prediction of the motion. A procedure for calculating the value of the game and for constructing minimax and maximin control strategies, which is convenient for numerical implementation, is obtained from this treatment and from the construction of hulls, convex upwards, of auxiliary functions from the method of stochastic program synthesis. The results of a numerical experiment are presented.

This paper is related to the investigations in [1–9].

1. FORMULATION OF THE PROBLEM

Suppose a system with a delay is described by the equation

$$dx(t)/dt = A(t)x(t) + A_h(t)x(t-h) + B(t)u + C(t)v, \quad t_0 \leqslant t \leqslant \vartheta \tag{1.1}$$

Here, x is an n-dimensional phase vector, u is an r-dimensional control vector (or the action of the first player), υ is an s-dimensional disturbance vector (or the action of the second player), $A(t)$, $A_h(t)$, $B(t)$ and $C(t)$ are continuous matrix functions, the delay $h = \text{const} > 0$, and t_0 and ϑ are fixed instants of time ($t_0 < \vartheta$). At the instant t_0, the initial state, $x_0[t_0 - h[\cdot]t_0] = \{x_0[\tau], t_0 - h \leqslant \tau \leqslant t_0\}$, of system (1.1) is known, where $x_0[\tau]$ is a piecewise-continuous vector function (that is, $x_0[\tau]$ can have a finite number of points of discontinuity of the first kind and, at these points, it is continuous to the right). The Borel measurable finite samples $u[t_0[\cdot]\vartheta) = \{u[t], t_0 \leqslant t < \vartheta\}$ and $\upsilon[t_0[\cdot]\vartheta) = \{\upsilon[t], t_0 \leqslant t < \vartheta\}$ are permissible. From an initial state $x_0[t_0 - h[\cdot]t_0]$, these samples uniquely [10, p. 19] generate the motion $x_0[t_0 - h[\cdot]\vartheta] = \{x[t], t_0 - h \leqslant t \leqslant \vartheta\}$ of system (1.1) ($x[t]$ is an absolutely continuous vector function when $t_0 \leqslant t \leqslant \vartheta$, which satisfies the condition $x[t] = x_0[t]$ when $t_0 - h \leqslant t \leqslant t_0$ and equality (1.1) for almost all t and $u = u[t]$, $\upsilon = \upsilon[t]$.

The natural number $N \geqslant 1$, the instants of time $t^{(i)} \in [t_0, \vartheta]$ ($t^{(i)} < t^{(i+1)}$, $i = 1, \ldots, N-1$, $t^{(N)} = \vartheta$), the constant $(p^{(i)} \times n)$-matrices $D^{(i)}$, where $1 \leqslant p^{(i)} \leqslant n$, and the n-dimensional vectors are specified.

It is required to find the control (or disturbance) aimed at minimizing (maximizing) the performance index

$$\gamma = \gamma(x[t_0[\cdot]\vartheta],\ u[t_0[\cdot]\vartheta),\ v[t_0[\cdot]\vartheta)) =$$

$$= \left(\sum_{i=1}^{N} |D^{(i)}(x[t^{(i)}] - g^{(i)})|^2 \right)^{1/2} + \int_{t_0}^{\vartheta} [\langle u[t],\ \Phi(t)u[t]\rangle - \langle v[t],\ \Psi(t)v[t]\rangle] dt \tag{1.2}$$

Here and below $|\cdot|$ is the Euclidean norm, $\langle\cdot,\cdot\rangle$ represents the scalar product of vectors, $\Phi(t)$ and $\Psi(t)$ are symmetric, continuous matrix functions and the quadratic forms $\langle u, \Phi(t)u\rangle$ and $\langle \upsilon, \Psi(t)\upsilon\rangle$ are positive definite when $t_0 \leqslant t \leqslant \vartheta$.‡

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‡For $N = 1$, this problem is considered in the following papers: STIKHINA, T. K., The theory of positional control in systems with a delay, Sverdlovsk, 1984: Deposited in the All-Union Institute for Scientific and Technical Information (VINITI) 06.04.84, No. 2051 and STIKHINA, T. K., Problems of positional control and modelling in dynamical systems. Candidate dissertation, 01.01.02. Sverdlovsk, 1985.

Remark. The performance index γ from (1.2) can be given from the start or the functional

$$\gamma_* = \left(\int_{t_0}^{\vartheta} |D(t)(x[t]-g(t))|^2\,dt\right)^{1/2} + \int_{t_0}^{\vartheta} [\langle u[\tau],\ \Phi(\tau)u[\tau]\rangle - \langle v[\tau],\ \Psi(\tau)v[\tau]\rangle]d\tau$$

is introduced as an approximating functional for the performance index, where $D(t)$ is a known piecewise-smooth matrix function and $g(t)$ is a specified piecewise-continuous vector function.

These two problems can be combined into a two-person differential game. Suppose the history of the motion $x[t_0-h[\cdot]t] = \{x[\tau], t_0-h \leq \tau < t\}$ has been put together under the action of the permissible forms $u[t_0[\cdot]t) = \{u[\tau], t_0 \leq \tau < t\}$ and $\upsilon[t_0[\cdot]t) = \{\upsilon[\tau], t_0 \leq \tau < t\}$ by the time t. The triple $\{x[\tau], t_0 - h[\cdot]t]), \upsilon[t_0[\cdot]t])\}$ is called the form of the control process by the time t. Following the concepts in [1–6], we shall say that the functional $\rho^0(x[t_0-h[\cdot]t_*], u[t_0[\cdot]t_*), \upsilon[t_0[\cdot]t_*))$, where t_* is the initial instant of time ($t_0 \leq t_* < \vartheta$), is the value and the pair of strategies which are the vector functions from the history of the motion $u(x[t_0-h[\cdot]t], \varepsilon)$ and $\upsilon(x[t_0-h[\cdot]t], \varepsilon)$, where t is the actual instant of time and ε is an accuracy parameter (see, for example [5, p. 68]), forms a saddle point in the game for system (1.1) with the performance index γ from, (1.2) if, whatever has served as the initial form of the process $\{x[t_0-h[\cdot]t_*], u[t_0[\cdot]t_*), \upsilon[t_0[\cdot]t_*)\}$, for any number $\zeta > 0$ a number $\varepsilon_* > 0$ and a function $\delta(\varepsilon) > 0$, $0 < \varepsilon \leq \varepsilon_*$ are found such that, on the one hand, for any number $\varepsilon > 0$, $\varepsilon \leq \varepsilon_*$, partitioning $\Delta = \Delta_k\{t_j\} = \{t_j{:}t_1 = t_*, t_j < t_{j+1}, j = 1,\ldots,k, t_{k+1} = \vartheta\}$ with step $\delta_k = \max_{j=1,\ldots,k}(t_{j+1}-t_j) \leq \delta(\varepsilon)$ and permissible form of the noise $\upsilon[t_*[\cdot]\vartheta)$ for the motion $x^{\varepsilon}_{\Delta,u^0}[t_0-h[\cdot]\vartheta]$, which is the solution of the step-by-step equation

$$dx^{\varepsilon}_{\Delta,u^0}[t]/dt = A(t)x^{\varepsilon}_{\Delta,u^0}[t] + A_h(t)x^{\varepsilon}_{\Delta,u^0}[t-h] + B(t)u^0(x^{\varepsilon}_{\Delta,u^0}[t_0-h[\cdot]t_j],\varepsilon) + C(t)v[t] \tag{1.3}$$

$$t_j \leq t < t_{j+1},\quad j=1,\ldots,k,\quad x^{\varepsilon}_{\Delta,u^0}[\tau] = x[\tau],\quad t_0-h \leq \tau \leq t_1 = t_*$$

the following inequality will hold

$$\gamma(x^{\varepsilon}_{\Delta,u^0}[t_0[\cdot]\vartheta],\ u^{\varepsilon}_{\Delta,u^0}[t_0[\cdot]\vartheta),\ v[t_0[\cdot]\vartheta)) \leq \rho^0(x[t_0-h[\cdot]t_*],\ u[t_0[\cdot]t_*),\ v[t_0[\cdot]t_*)) + \zeta \tag{1.4}$$

where $u^{\varepsilon}_{\Delta,u0}[t_0[\cdot]\vartheta) = \{u^{\varepsilon}_{\Delta,u0}[t] = u[t], t_0 \leq t < t_*, u^{\varepsilon}_{\Delta,u0}[t] = u^0(x^{\varepsilon}_{\Delta,u0}[t_0-h[\cdot]t_j], \varepsilon), t_j \leq t < t_{j+1}, j = 1,\ldots,k\}$

On the other hand, the inequality

$$\gamma(x^{\varepsilon}_{\Delta,v^0}[t_0[\cdot]\vartheta),\ u[t_0[\cdot]\vartheta),\ v^{\varepsilon}_{\Delta,v^0}[t_0[\cdot]\vartheta)) \geq \rho^0(x[t_0-h[\cdot]t_*],\ u[t_0[\cdot]t_*),\ v[t_0[\cdot]t_*)) - \zeta \tag{1.5}$$

where $\upsilon^{\varepsilon}_{\Delta,\upsilon0}[t_0[\cdot]\vartheta) = \{\upsilon^{\varepsilon}_{\Delta,\upsilon0}[t] = \upsilon[t], t_0 \leq t < t_*, \upsilon^{\varepsilon}_{\Delta,\upsilon0}[t] = \upsilon^0(x^{\varepsilon}_{\Delta,\upsilon0}[t_0-h[\cdot]t_j], \varepsilon), t_j \leq t < t_{j+1}, j = 1,\ldots,k\}$, will hold for any number $\varepsilon > 0$, $\varepsilon \leq \varepsilon_*$ and partitioning $\Delta = \Delta_k\{t_j\}$ with a step size $\delta_k \leq \delta(\varepsilon)$ and a permissible form of the control $u[t_*[\cdot]\vartheta)$ for the motion $x^{\varepsilon}_{\Delta,\upsilon0}[t_0-h[\cdot]\vartheta)$, which is a solution of the step-by-step equation (1.3), where $x^{\varepsilon}_{\Delta,u0}[t]$ has to be replaced by $x^{\varepsilon}_{\Delta,\upsilon0}[t]$, $u^0(x^{\varepsilon}_{\Delta,u0}[t_0-h[\cdot]t_j], \varepsilon)$ by $u[t]$ and $\upsilon^0(x^{\varepsilon}_{\Delta\upsilon0}[t_0-h[\cdot]t_j]$, by $\varepsilon)$ respectively.

We will call the strategies $u^0(\cdot)$ and $\upsilon^0(\cdot)$ the optimal minimax strategy and the optimal maximin strategy respectively. On developing the reasoning in accordance with the scheme from [1, 5, 6] and taking account of the results of [2–4], it can be shown that, in the game being considered, the value $\rho^0(x[t_0-h[\cdot]t_*], u[t_0[\cdot]t_*), \upsilon[t_0[\cdot]t_*))$ and the optimal strategies $u^0(x[t_0-h[\cdot]t], \varepsilon)$ and $\upsilon^0(x[t_0-h[\cdot]t_*], \varepsilon)$ exist. Here, the actions $u^0(x^{\varepsilon}_{\Delta u0}[t_0-h[\cdot]t_j], \varepsilon)$ and $\upsilon^0(x^{\varepsilon}_{\Delta\upsilon0}[t_0-h[\cdot]t_j], \varepsilon)$ from (1.3)–(1.5) are formed using the value functional $\rho^0(\cdot)$ by the method of extremal shift (see, for example, [6, p. 150]) in the appropriate accompanying motions.

We emphasize that the optimal strategies $u^0(\cdot)$ and $\upsilon^0(\cdot)$ only use information on the history of the system.

The calculation of the value $\rho^0(\cdot)$ and the construction of the strategies $u^0(\cdot)$ and $\upsilon^0(\cdot)$ in differential game (1.1), (1.2) make up the main content of this paper.

A functional treatment of the control process being considered is given in the next section. This treatment reduces the initial problem concerning the control of a system with delay (1.1) on a minimax (maximin) performance index (1.2) to an auxiliary problem on the minimax (maximin) control of a system without delay with a terminal dimensionality than the dimensionality n of the vector x. This functional interpretation starts from the constructions proposed in [11, p. 150] when investigating problems of the stability of motions in systems with delay and which are employed, in particular, in [9] for problems of the control of ordinary differential systems with a functional value. The above information enables one to use the results and to transform the constructions proposed in [5–9] for problems of the game control of systems without delay as it applies to the problem under consideration here.

2. FUNCTIONAL TREATMENT

Suppose that $F(\xi, \tau)$ is an $n \times n$ matrix which satisfies the following conditions: $F(\xi, \tau)$ when $\xi < \tau$; $F(\tau, \tau) = E^{[n]}$, where $E^{[n]}$ is an n-dimensional unit matrix; $dF(\xi, \tau)/d\xi = A(\xi)F(\xi, t) + A_h(\xi)F(\xi - h, \tau)$ when $\xi > \tau$. Then [10, p. 64], the motion $x[t_0 - h[\cdot]\vartheta]$ which extends the initial history $x_*[t_0 - h[\cdot]t_*]$, $t_0 \leq t_* < \vartheta$) by virtue of Eq. (1.1) under the action of the permissible forms $u[t_*[\cdot]\vartheta)$ and $\upsilon[t_*[\cdot]\vartheta)$ can be written using the Cauchy formula

$$
\begin{aligned}
&x[\tau] = x_*[\tau], \quad t_0 - h \leq \tau < t_* \\
&x[t] = F(t, t_*)x_*[t_*] + \int_{t_*}^{t_*+h} F(t,\tau)A_h(\tau)x_*[\tau - h]d\tau + \\
&+ \int_{t_*}^{t} F(t,\tau)(B(\tau)u[\tau] + C(\tau)\nu[\tau])d\tau, \quad t_* \leq t \leq \vartheta
\end{aligned}
\tag{2.1}
$$

Suppose a history $x[t_0 - h[\cdot]t]$ of system (1.1) has been realized by the time $t \in [t_0, \vartheta]$. We shall call the p-dimensional vector

$$
\mathbf{y}(x[t_0 - h[\cdot]t]) = \{y^{(1)}[t], \ldots, y^{(N)}[t]\}, \quad p = p^{(1)} + \ldots + p^{(N)} \tag{2.2}
$$

where

$$
y^{(i)}[t] = \begin{cases} D^{(i)}(x[t^{(i)}] - g^{(i)}), & t^{(i)} \leq t \\ D^{(i)}\left(F(t^{(i)}, t)x[t] + \int_t^{t+h} F(t^{(i)}, \tau)A_h(\tau)x[\tau - h]d\tau - g^{(i)}\right), & t < t^{(i)} \end{cases}
$$

the information object which corresponds to this history. The notation in (2.2) means that the first $p^{(1)}$ components of the vector $\mathbf{y}(x[t_0 - h[\cdot]$ are identical with the components of the $p^{(1)}$-dimensional vector $y^{(1)}[t]$, the following $p^{(2)}$ components are identical with the components of the $p^{(2)}$-dimensional vector $y^{(2)}[t]$, and so on. The last $p^{(N)}$ components of $\mathbf{y}(x[t_0 - h[\cdot]t)$ are identical with the vector $y^{(N)}[t]$. Moreover, if $N = 1$, then $\mathbf{y}(x[t_0 - h[\cdot]t]) = y^{(N)}[t], p = p^{(N)}$.

We use the notation

$$
y^*[t] = y^*(u[t_0[\cdot]t), \nu[t_0[\cdot]t)) = \int_{t_0}^{t} [\langle u[\tau], \Phi(\tau)u[\tau]\rangle - \langle \nu[\tau], \Psi(\tau)\nu[\tau]\rangle]d\tau \tag{2.3}
$$

The performance index γ from (1.2) can now be written in the form

$$
\gamma = |\mathbf{y}(x[t_0 - h[\cdot]\vartheta])| + y^*[\vartheta] \tag{2.4}
$$

Next, we use the notation

$$
B_*^{(i)}(t) = D^{(i)}F(t^{(i)}, t)B(t), \quad C_*^{(i)}(t) = D^{(i)}F(t^{(i)}, t)C(t), \quad i = 1, \ldots, N
$$

From the ($p^{(i)} \times r$-matrices $B_*^{(i)}(t)$ ($i = 1, \ldots, N$), we set up the $p \times r$ matrix $\mathbf{B}_*(t)$ such that the first $p^{(1)}$ rows of the matrix $\mathbf{B}_*(t)$ are identical with the rows of the matrix $B_*^{(1)}(t)$, the following $p^{(2)}$ rows are identical with the rows of $B_*^{(2)}(t)$, and so on, and the last $p^{(N)}$ rows of the matrix $\mathbf{B}_*(t)$ are identical with the rows of the matrix $B_*^{(N)}(t)$. Using the same rule, we set up the ($p \times s$)-matrix $C_*(t)$ from the ($p^{(i)} \times s$)-matrices $C_*^i(t)$.

We now introduce an auxiliary $\mathbf{w}$-system. Suppose $\mathbf{w}$ is the p-dimensional phase vector of this system and its evolution obeys the equation without delay

$$
d\mathbf{w}/dt = \mathbf{B}_*(t)u + \mathbf{C}_*(t)\nu, \quad t_0 \leq t \leq \vartheta \tag{2.5}
$$

For an initial state $\mathbf{w}[t_0]$, specified at the instant t_0, the permissible forms $u[t_0[\cdot]\vartheta)$ and $\upsilon[t_0[\cdot]\vartheta)$

generate the absolutely continuous motion $\mathbf{w}[t_0\vartheta]\vartheta] = \{\mathbf{w}[t], t_0 \leqslant t \leqslant \vartheta\}$ of the w-system (2.5). We will estimate the quality of this motion using the performance index

$$\gamma_{\mathbf{w}} = |\mathbf{w}[\vartheta]| + y^*[\vartheta] \tag{2.6}$$

where $y^*[\vartheta]$ from (2.3) when $t = \vartheta$.

The following lemma, which establishes the link between the evolution, by virtue of system (1.1), of the information object $\mathbf{y}(x[t_0 - h[\cdot]t$ from (2.2) and the evolution of the w-system (2.5), holds.

Lemma 1. Whatever the initial history $x[t_0 - h[\cdot]t_*]$ of system (1.1) ($t_0 \leqslant t_* \leqslant \vartheta$), the instant $t^* \in (t_*, \vartheta)$, the permissible forms $u[t_*[\cdot]t^*) = \{u[t], t_* \leqslant t < t^*\}$ of the control and $\upsilon[t_*[\cdot]t^*) = \{\upsilon[t], t_* \leqslant t < t^*\}$ of the disturbance, the equality

$$\mathbf{w}[t^*] = \mathbf{y}(x[t_0 - h[\cdot]t^*])$$

where $\mathbf{y}(x[t_0 - h[\cdot]t^*])$ is the information object (2.2) corresponding to the history $x[t_0 - h[\cdot]t^*]$, will hold for the history $x[t_0 - h[\cdot]t^*]$, which is realized by the instant of time t^* accompanying the motion of system (1.1) from the state $x[t_* - h[\cdot]t_*]$ under the action of these controls $u[t_*[\cdot]t^*)$ and $\upsilon[t_*[\cdot]t^*)$ and for the state $\mathbf{w}[t^*]$ which is realized by the time t^* in the motion of the w-system (2.5) from the state $\mathbf{w}[t_*] = \mathbf{y}(x[t_0 - h[\cdot]t_*])$ (see (2.2)) when $t = t_*$) under the action of the same controls.

The validity of Lemma 1 can be verified directly using Cauchy's formula (2.1) and the properties of the matrix $F(\xi, \tau)$.

For the w-system (2.5) and the performance index $\gamma_{\mathbf{w}}$ from (2.6), we consider a two-person differential game (the actions u of the first player are aimed at minimizing $\gamma_{\mathbf{w}}$ while the aims of the second player are aimed at maximizing $\gamma_{\mathbf{w}}$).

It is well known [5, 6] that such a game has a value $\rho^0{}_{\mathbf{w}}(t, \mathbf{w}, y^*)$ and a saddle point which is made up from the optimal minimax $u^0{}_{\mathbf{w}}(t, \mathbf{w}, \varepsilon)$ and maximin $\upsilon^0_{\mathbf{w}}(t, \mathbf{w}, \varepsilon)$ strategies. This means that, whatever the initial position $\{t_*, \mathbf{w}_* = \mathbf{w}[t_*], y^*_* = y^*[t_*]\}$, ($t_0 \leqslant t_* < \vartheta$) and for any number $\zeta > 0$ which has been specified beforehand, the step-by-step control law which forms the actions

$$u[t] = u^0_{\mathbf{w}}(t_j, \mathbf{w}[t_j], \varepsilon), \quad t_j \leqslant t < t_{j+1}, \quad j = 1, \ldots, k, \quad t_1 = t_*, \quad t_{k+1} = \vartheta \tag{2.7}$$

guarantees the inequality

$$\gamma_{\mathbf{w}} \leqslant \rho^0_{\mathbf{w}}(t_*, \mathbf{w}_*, y^*_*) + \zeta \tag{2.8}$$

regardless of the dependence on the disturbance which is realized, if the values of the parameter $\varepsilon > 0$ and the step $\delta_k = \max_{j=1,\ldots,k}(t_{j+1} - t_j)$ are chosen to be sufficiently small. On the other hand, a step law which prescribes the disturbance actions

$$\upsilon[t] = \upsilon^0_{\mathbf{w}}(t_j, \mathbf{w}[t_j], \varepsilon), \quad t_j \leqslant t < t_{j+1}, \quad j = 1, \ldots, k, \quad t_1 = t_*, \quad t_{k+1} = \vartheta \tag{2.9}$$

$$\gamma_{\mathbf{w}} \geqslant \rho^0_{\mathbf{w}}(t_*, \mathbf{w}_*, y^*_*) - \zeta \tag{2.10}$$

subject to the condition that $\varepsilon > 0$ and that $\delta_k = \max_{j=1,\ldots,k}(t_{j+1} - t_j)$ are sufficiently small, regardless of the control which is realized.

The optimal strategies $u^0{}_{\mathbf{w}}(t, \mathbf{w}, \varepsilon)$ and $\upsilon^0{}_{\mathbf{w}}(t, \mathbf{w}, \varepsilon)$ are constructed as extremal strategies to the value function $u^0(x[t_0 - h[\cdot]t], \varepsilon) = u^0{}_{\mathbf{w}}(t, \mathbf{y}(x[t_0 - h[\cdot]t], \varepsilon)$ (see [5, pp. 210 and 220]).

It follows from the properties (2.7)–(2.10) of the value $\rho^0_{\mathbf{w}}(\cdot)$ and the optimal strategies $u^0_{\mathbf{w}}(\cdot)$ and $\upsilon^0_{\mathbf{w}}(\cdot)$ of the auxiliary game (2.5), (2.6), if the notation (2.3), the representation (2.4) and Lemma 1 are taken into account, that the functional $\rho^0_{\mathbf{w}}(t_*, \mathbf{y}(x[t_0 - h[\cdot]t_*]), y^*[t_*])$ and the vector functions $u^0_{\mathbf{w}}(t, \mathbf{y}(x[t_0 - h[\cdot]t]), \varepsilon)$ and $\upsilon^0_{\mathbf{w}}(t, \mathbf{y}(x[t_0 - h[\cdot]t]), \varepsilon)$ will satisfy requirements (1.3)–(1.5) from section 1. The following assertion therefore holds.

Theorem 1. Whatever the possible form $\{x[t_0 - h[\cdot]t_*], u[t_0[\cdot]t_*), \upsilon[t_0[\cdot]t_*)\}$ of the control process (1.1), (1.2) by the instant of time $t_* \in [t_0, \vartheta]$, the equality

$$\rho^0(x[t_0 - h[\cdot]t_*], \ u[t_0[\cdot]t_*), \ \upsilon[t_0[\cdot]t_*)) = \rho^0_{\mathbf{w}}(t_*, \mathbf{y}(x[t_0 - h[\cdot]t_*]), \ y^*[t_*])$$

holds for the value of game (1.1).

Here, the strategies $u^0(x[t_0 - h[\cdot]t], \varepsilon) = u^0_w(t, y(x[t_0 - h[\cdot]t], \varepsilon)$ and $\upsilon^0(x[t_0 - h[\cdot]t], \varepsilon) = \upsilon^0_w(t, y(x[t_0 - h[\cdot]t], \varepsilon), t_* \leq t < \vartheta$ form the saddle point in game (1.1), (1.2).

Remark. Since, as was pointed out above, the function $\rho^0_w(\cdot)$ and the strategies $u^0_w(\cdot)$ and $\upsilon^0_w(\cdot)$ exist, Theorem 1 independently establishes the existence of a value $\rho^0(\cdot)$ and the optimal strategies $u^0_w(\cdot)$ and $\upsilon^0_w(\cdot)$ in the differential game (1.1), (1.2) which is being considered.

3. CALCULATION OF THE VALUE OF THE GAME

According to Theorem 1, in order to determine the functional $\rho^0(\cdot)$ of the value of the initial differential game (1.1), (1.2), it is sufficient to construct the value function $\rho^0_w(\cdot)$ of the auxiliary differential game (2.5), (2.6). In order to do this, we make use of the method of hulls, which are convex upwards, of deterministic functions which arise in the construction of stochastic program synthesis [5].

Suppose some position $\{t_*, \mathbf{w}_* = \mathbf{w}[t_*], y^*_* = y^*[t_*]\}$, $t_0 \leq t_* < \vartheta$ has occurred. We fix the partitioning

$$\Delta_k = \Delta_k\{\tau_j\} = \{\tau_j:\ \tau_1 = t_*,\ \tau_j < \tau_{j+1},\ j = 1,\dots,k,\ \tau_{k+1} = \vartheta\} \tag{3.1}$$

of the time interval $[t_*, \vartheta]$. Suppose $\mathbf{l} = \{l_1, \dots, l_p\}$ is a p-dimensional vector. We use the notation

$$\Delta\psi_j(t_*, \mathbf{l}) = \int_{\tau_j}^{\tau_{j+1}} \max_{\upsilon \in R^s} \min_{u \in R^r} [\langle \mathbf{l}, (\mathbf{B}_*(\tau)u + \mathbf{C}_*(\tau)\upsilon)\rangle + \langle u, \Phi(\tau)u\rangle - \langle \upsilon, \Psi(\tau)\upsilon\rangle]d\tau =$$

$$= \int_{\tau_j}^{\tau_{j+1}} \langle \mathbf{l}, \mathbf{M}(\tau)\mathbf{l}\rangle d\tau \tag{3.2}$$

where

$$\mathbf{M}(\tau) = \frac{1}{4}[\mathbf{C}_*(\tau)\Psi^{-1}(\tau)\mathbf{C}^T_*(\tau) - \mathbf{B}_*(\tau)\Phi^{-1}(\tau)\mathbf{B}^T_*(\tau)] \tag{3.3}$$

The superscript T denotes transposition. The last equality in (3.2) follows from the direct calculation of the maximin. In (3.3), $\Phi^{-1}(\tau)$ and $\Psi^{-1}(\tau)$ are the inverse matrices of $\Phi(\tau)$ and $\Psi(\tau)$, respectively.

We write the magnitude of the program extremum

$$e_w(t_*, \mathbf{w}_*, y^*_*;\ \Delta_k) = \max_{\|\mathbf{l}\| \leq 1}[\langle \mathbf{l}, \mathbf{w}_*\rangle + \varphi_1(t_*, \mathbf{l})] + y^*_* \tag{3.4}$$

The function $\varphi_1(t_*, \mathbf{l})$ is determined from the sequence of functions $\varphi_j(t_*, \mathbf{l})$ which satisfy the following recurrence relations

$$\varphi_{k+1}(t_*, \mathbf{l}) \equiv 0, \quad \psi_j(t_*, \mathbf{l}) = \Delta\psi_j(t_*, \mathbf{l}) + \varphi_{j+1}(t_*, \mathbf{l}) \tag{3.5}$$

$$\varphi_j(t_*, \mathbf{l}) = \{\psi_j(t_*, \cdot)\}^*, \quad j = 1,\dots,k \tag{3.6}$$

In (3.6), the symbol $\{\psi(\cdot)\}^*$ denotes the hull, which is convex upwards, of the function $\psi(\mathbf{l})$ in the domain $\{\mathbf{l}: |\mathbf{l}| \leq 1\}$, that is, the function which is the minimum function of all concave functions which majorize $\psi(\mathbf{l})$, $|\mathbf{l}| \leq 1$.

The equality [7]

$$\rho^0_w(t_*, \mathbf{w}_*, y^*_*) = \lim_{k \to \infty} e_w(t_*, \mathbf{w}_*, y^*_*;\ \Delta_k) \tag{3.7}$$

holds for any sequence of partitioning $\Delta_k = \Delta_k\{\tau_j\}$ (3.1), which satisfies the condition $\delta_k = \max_{j = 1,\dots,k} (\tau_{j+1} - \tau_j) \to 0$ as $k \to \infty$.

Following [8], we construct the functions $\varphi_j(t_*, \mathbf{l})$, $|\mathbf{l}| \leq 1, j = 1, \dots, k$. We put

$$\mathbf{K}(\tau) = \int_{\tau}^{\vartheta} \mathbf{M}(\xi)d\xi, \quad t_* \leq \tau \leq \vartheta \tag{3.8}$$

In view of (3.3), $\mathbf{K}(\tau)$ is a symmetric $p \times p$ matrix-function. An orthonormal basis $\{\mathbf{l}_q^{[j]}, q = 1, \ldots, p\}$ of a p-dimensional Euclidean space, composed of the eigenvectors of the matrix $\mathbf{K}(\tau_j)$, is found for each $j = 1, \ldots, k$. Quadratic $p \times p$ matrices $\mathbf{Q}_j$ ($\mathbf{Q}_j^{-1} = \mathbf{Q}_j^T$) ($j = 1, \ldots, k$) are constructed from the column vectors $\{\mathbf{l}_q^{[j]}, q = 1, \ldots, p\}$. These matrices define orthogonal transformations of the variables

$$\mathbf{l} = \mathbf{Q}_j\tilde{\mathbf{l}}, \quad |\mathbf{l}| = |\mathbf{Q}_j\tilde{\mathbf{l}}| = |\tilde{\mathbf{l}}| \tag{3.9}$$

which reduces the quadratic forms $\langle \mathbf{l}, \mathbf{K}(\tau_j)\mathbf{l} \rangle$ to the canonical form

$$\langle \mathbf{l}, \mathbf{K}(\tau_j)\mathbf{l} \rangle = \langle \mathbf{Q}_j\tilde{\mathbf{l}}, \mathbf{K}(\tau_j)\mathbf{Q}_j\tilde{\mathbf{l}} \rangle = \lambda_1^{[j]}\tilde{l}_1^2 + \ldots + \lambda_p^{[j]}\tilde{l}_p^2 \tag{3.10}$$

where the real numbers $\lambda_q^{[j]}$ ($q = 1, \ldots, p$) are the eigenvalues of the matrix $\mathbf{K}(\tau_j)$ to which the eigenvectors $\mathbf{l}_q^{[j]}$ correspond.

We now put

$$\lambda_j^* = \max\left\{0, \max_{m=j,\ldots,k} \max_{q=1,\ldots,p} \lambda_q^{[m]}\right\}, \quad j = 1, \ldots, k \tag{3.11}$$

When $j = k$, in accordance with (3.2), (3.5) and (3.8), we obtain

$$\psi_k(t_*, \mathbf{l}) = \langle \mathbf{l}, \mathbf{K}(\tau_k)\mathbf{l} \rangle \tag{3.12}$$

The equality

$$\psi_k(t_*, \mathbf{l}) = \psi_k(t_*, \mathbf{Q}_k\tilde{\mathbf{l}}) = \lambda_1^{[k]}\tilde{l}_1^2 + \ldots + \lambda_p^{[k]}\tilde{l}_p^2 \tag{3.13}$$

follows from (3.10) and (3.12).

We put

$$\lambda^* = \max_{q=1,\ldots,p} \lambda_q^{[k]} \tag{3.14}$$

If $\lambda^* \leqslant 0$, the function $\psi_k(t_*, \mathbf{l})$ is a negative quadratic form of constant sign and, consequently, a concave function and the equality

$$\varphi_k(t_*, \mathbf{l}) = \{\psi_k(t_*, \cdot)\}^* = \psi_k(t_*, \mathbf{l}), \quad |\mathbf{l}| \leqslant 1 \tag{3.15}$$

holds.

Suppose $\lambda^* > 0$. A vector is found in the basis $\{\mathbf{l}_q^{[k]}, q = 1, \ldots, p\}$ which corresponds to the number λ^*, which we denote by $\mathbf{l}^*$. On numbering the eigenvalues $\lambda^{[k]}$ in the proper manner, we assume that $\lambda^* = \lambda^*_p$, $= \mathbf{l}^* = \mathbf{l}_p^{[k]}$ and construct the function

$$\begin{aligned}\varphi^*(\mathbf{l}) &= \varphi^*(\mathbf{Q}_k\tilde{\mathbf{l}}) = \lambda_1^{[k]}\tilde{l}_1^2 + \ldots + \lambda_{p-1}^{[k]}\tilde{l}_{p-1}^2 + \lambda^*(1 - \tilde{l}_1^2 - \ldots - \tilde{l}_{p-1}^2) = \\ &= \psi_k(t_*, \mathbf{Q}_k\tilde{\mathbf{l}}) - \lambda^*|\tilde{\mathbf{l}}|^2 + \lambda^* = \psi_k(t_*, \mathbf{l}) - \lambda^*|\mathbf{l}|^2 + \lambda^*\end{aligned} \tag{3.16}$$

It can be verified [8] that, when $\lambda^* > 0$, the function $\varphi^*(\mathbf{l})$ is the required hull, which is convex upwards, of the function $\psi_k(t_*, \mathbf{l})$.

Hence, when $\lambda^* > 0$, the equality

$$\varphi_k(t_*, \mathbf{l}) = \{\psi_k(t_*, \cdot)\}^* = \varphi^*(\mathbf{l}), \quad |\mathbf{l}| \leqslant 1 \tag{3.17}$$

holds.

Using (3.11)–(3.14), we obtain that the function $\varphi_k(t_*, \mathbf{l})$, both in the case when $\lambda^* \leqslant 0$ (see (3.15)) and when $\lambda^* > 0$ (see (3.16), (3.17)), is defined by the expression

$$\varphi_k(t_*, \mathbf{l}) = \langle \mathbf{l}, \mathbf{K}(\tau_k)\mathbf{l} \rangle - \lambda_k^*|\mathbf{l}|^2 + \lambda_k^* \tag{3.18}$$

We now consider an induction with respect to j from $j = k$ to $j = 1$.

We assume that, at the $(j + 1)$th step, the required hull has already been constructed and has the form

$$\varphi_{j+1}(t_*,\mathbf{l}) = \langle \mathbf{l}, \mathbf{K}(\tau_{j+1})\mathbf{l}\rangle - \lambda^*_{j+1}\,|\mathbf{l}|^2 + \lambda^*_{j+1} \tag{3.19}$$

Equality (3.19) holds in every case when $j + 1 = k$ which follows from (3.18).
The equality

$$\varphi_j(t_*,\mathbf{l}) = \langle \mathbf{l}, \mathbf{K}(\tau_j)\mathbf{l}\rangle - \lambda^*_j\,|\mathbf{l}|^2 + \lambda^*_j \tag{3.20}$$

then holds for the function $\varphi_j(t_*, \mathbf{l})$ from (3.5) and (3.6).

To prove equality (3.20), using (3.8) and (3.19) we calculate

$$\psi_j(t_*,\mathbf{l}) = \langle \mathbf{l}, \mathbf{K}(\tau_j)\mathbf{l}\rangle - \lambda^*_{j+1}\,|\mathbf{l}|^2 + \lambda^*_{j+1} \tag{3.21}$$

in accordance with (3.2) and (3.5).
On taking (3.9) and (3.10) into account, we have

$$\psi_j(t_*,\mathbf{l}) = \psi_j(t_*,\mathbf{Q}_j\tilde{\mathbf{l}}) = \lambda_1^{[j]}\tilde{l}_1^2 + \ldots + \lambda_p^{[j]}\tilde{l}_p^2 - \lambda^*_{j+1}\,|\tilde{\mathbf{l}}|^2 + \lambda^*_{j+1} \tag{3.22}$$

We put

$$\lambda^* = \max_{q=1,\ldots,p} \lambda_q^{[j]} \tag{3.23}$$

When $\lambda^* \leqslant \lambda^*_{j+1}$ (see (3.11)), the function $\psi_j(t_*, \mathbf{l})$ from (3.21) and (3.22) turns out to be concave and, consequently, the relation

$$\varphi_j(t_*,\mathbf{l}) = \{\psi_j(t_*,\cdot)\}^* = \psi_j(t_*,\mathbf{l}), \quad \|\mathbf{l}\| \leqslant 1 \tag{3.24}$$

holds
In the case when $\lambda^* > \lambda^*_{j+1}$, we set

$$\varphi_j(t_*,\mathbf{l}) = \langle \mathbf{l}, \mathbf{K}(\tau_j)\mathbf{l}\rangle - \lambda^*\,|\mathbf{l}|^2 + \lambda^* \tag{3.25}$$

As in the case when $j = k$, it can be verified directly that the function $\varphi_j(t_*, \mathbf{l})$ from (3.25) actually is the required hull, which is convex upwards, of the function $\psi_j(t_*, \mathbf{l})$ from (3.21) when $\lambda^* > \lambda^*_{j+1}$.
We conclude from (3.11), (3.21) and (3.23) that, in both cases, expressions (3.24) and (3.25) can be written with the same formula (3.20).

By virtue of the induction, equality (3.2) holds for any $j = 1, \ldots, k$. This completes the construction of the sequence of functions $\varphi_j(t_*, \mathbf{l})$, $|\mathbf{l}| \leqslant 1, j = 1, \ldots, k$.
From (3.4) and (3.2), taking account of the fact that $\tau_1 = t_*$, we derive

$$e_{\mathbf{w}}(t_*, \mathbf{w}_*, y^*_*;\ \Delta_k) = \max_{\|\mathbf{l}\| \leqslant 1}[\langle \mathbf{l}, \mathbf{w}_*\rangle + \langle \mathbf{l}, \mathbf{K}(t_*)\mathbf{l}\rangle - \lambda^*_1\|\mathbf{l}\|^2] + \lambda^*_1 + y^*_* \tag{3.26}$$

We now consider the matrix $\mathbf{K}(\tau)$ from (3.8). Suppose $\lambda[\tau]$ is its largest eigenvalue. We put

$$\lambda^*_{t_*} = \max_{t_* \leqslant \tau \leqslant \vartheta} \lambda[\tau] = \lambda[\tilde{\tau}] \tag{3.27}$$

We shall assume that the number $\tilde{\tau} = \tilde{\tau}(t_*)$ has been included in the set of points τ_j of the partitioning Δ_k (see (3.1)) of the interval $[t_*, \vartheta]$. It then follows from (3.26), when account is taken of (3.11) and (3.27), that the quantity $e_{\mathbf{w}}(t_*, \mathbf{w}_*, y^*_*; \Delta_k)$ is independent of the choice of k, $k \geqslant 2$ and it is sufficient to take the partitioning $\Delta_2\{\tau_j\} = \{\tau_1 = t_*, \tau_2 = \tilde{\tau}, \tau_3 = \vartheta\}$ for its calculation. Consequently, according to (3.7), the equality

$$\rho^0_{\mathbf{w}}(t_*, \mathbf{w}_*, y^*_*) = \max_{\|\mathbf{l}\| \leqslant 1}[\langle \mathbf{l}, \mathbf{w}_*\rangle + \langle \mathbf{l}, \mathbf{K}(t_*)\mathbf{l}\rangle - \lambda^*_{t_*}\,|\mathbf{l}|^2] + \lambda^*_{t_*} + y^*_* \tag{3.28}$$

also holds and is valid for any possible position $\{t_*, \mathbf{w}_*, y^*_*\}$.
By virtue of Theorem 1, when $\mathbf{w}_* = \mathbf{y}(x[t_0 - h[\cdot]t_*])$ (see (2.2) in the case when $t = t_*$) and the notation (2.3) is taken into account, expression (3.28) gives a formula for calculating the value $\rho^0(x[t_0 - h[\cdot]t], \varepsilon)$ of the initial differential game (1.1), (1.2).

4. OPTIMAL STRATEGIES

In order to determine the optimal strategies $u^0(x[t_0 - h[\cdot]t], \varepsilon)$ and $\upsilon^0(x[t_0 - h[\cdot]t], \varepsilon)$ for the initial game (1.1), (1.2), following the material which has been presented above, we construct the strategies $u^0_{\mathbf{w}}(t, \mathbf{w}, \varepsilon)$ and $\upsilon^0_{\mathbf{w}}(t, \mathbf{w}, \varepsilon)$ which are optimal for the auxiliary game (2.5), (2.6).

Suppose that a position $\{t, \mathbf{w} = \mathbf{w}[t], y^* = y^*[t]\}$ has occurred and that a value of the parameter $\varepsilon > 0$ has been selected. The optimal actions $u^0_{\mathbf{w}}(t, \mathbf{w}, \varepsilon)$ and $\upsilon^0_{\mathbf{w}}(t, \mathbf{w}, \varepsilon)$ are formed by the method of an extremal shift to the corresponding points, which are determined using the function $\rho^0_{\mathbf{w}}(\cdot)$ from (3.28) (see the details in [5, p. 416]). We now present the resulting formulae

$$u^0_{\mathbf{w}}(t,\mathbf{w},\varepsilon) = -\frac{1}{2}\Phi^{-1}(t)\mathbf{B}^T_*(t)\mathbf{l}^0_u(t,\mathbf{w},\varepsilon) \tag{4.1}$$

$$\upsilon^0_{\mathbf{w}}(t,\mathbf{w},\varepsilon) = \frac{1}{2}\Psi^{-1}(t)\mathbf{C}^T_*(t)\mathbf{l}^0_\upsilon(t,\mathbf{w},\varepsilon)$$

where $\mathbf{l}^0_u = \mathbf{l}^0_u(t, \mathbf{w}, \varepsilon)$ and $\mathbf{l}^0_\upsilon = \mathbf{l}^0_\upsilon(t, \mathbf{w}, \varepsilon)$ are the maximizing vectors

$$\langle \mathbf{l}^0_u, \mathbf{w}\rangle + \langle \mathbf{l}^0_u, \mathbf{K}(t)\mathbf{l}^0_u\rangle - \lambda^*_t \,|\,\mathbf{l}^0_u\,|^2 - [(1+|\,\mathbf{l}^0_u\,|^2)\ (\varepsilon(t-t_0)+\varepsilon)]^{1/2} = \max_{\|\mathbf{l}\|\le 1}[\mathbf{Idem}(\mathbf{l}^0_u \to \mathbf{l})]$$

$$\langle \mathbf{l}^0_\upsilon, \mathbf{w}\rangle + \langle \mathbf{l}^0_\upsilon, \mathbf{K}(t)\mathbf{l}^0_\upsilon\rangle - \lambda^*_t \,|\,\mathbf{l}^0_\upsilon\,|^2 + [(1+|\,\mathbf{l}^0_\upsilon\,|^2)\ (\varepsilon(t-t_0)+\varepsilon)]^{1/2} = \max_{\|\mathbf{l}\|\le 1}[\mathbf{Idem}(\mathbf{l}^0_\upsilon \to \mathbf{l})] \tag{4.2}$$

Here, "Idem" on the right-hand side of the equality denotes an expression which is identical with the left-hand side of this equality on replacing the symbols in the brackets.

Note that the actions $u^0_{\mathbf{w}}(\cdot)$ and $\upsilon^0_{\mathbf{w}}(\cdot)$ are bounded.

On substituting $\mathbf{w} = \mathbf{y}(x[t_0 - h[\cdot]t])$ into (4.1) and (4.2) (see 2.2)), in accordance with Theorem 1 we obtain formulae which define the strategies $u^0(x[t_0 - h[\cdot]t], \varepsilon)$ and $\upsilon^0(x[t_0 - h[\cdot]t], \varepsilon)$, which are optimal in the differential game (1.1), (1.2).

5. EXAMPLE

Consider the system with delay

$$\begin{aligned}
&\ddot r_1(t) = -2r_1(t) - 0.4\dot r_1(t) + 0.02r_2(t) - r_1(t-1) - \\
&- 0.4\dot r_1(t-1) + 0.4r_2(t-1) - \dot r_2(t-1) + (5-t)u_1 + 2\upsilon_1 \\
&\ddot r_2(t) = 0.01r_1(t) - r_2(t) - 0.1\dot r_2(t) - 0.3r_1(t-1) + \\
&+0.7\dot r_1(t-1) - 0.4r_2(t-1) + 0.5\dot r_2(t-1) + (4-0.5t)u_2 + 3\upsilon_2 \\
&0 \le t \le 4, \quad r_1(\tau) = \sin\tau, \quad r_2(\tau) = \cos\tau, \quad -1 \le \tau \le 0
\end{aligned} \tag{5.1}$$

where a dot above a symbol denotes a derivative with respect to t.

Suppose the performance index of the control process has the form

$$\begin{aligned}
\gamma = {} & (r_1^2[0.5] + r_2^2[0.5] + \dot r_1^2[0.5] + \dot r_2^2[0.5] + r_1^2[1] + r_2^2[1] + r_1^2[1.5] + \\
& + \dot r_1^2[2] + \dot r_2^2[2.5] + r_2^2[3] + \dot r_1^2[3.5] + \dot r_2^2[3.5] + r_1^2[4] + r_2^2[4] + \\
& + \dot r_1^2[4] + \dot r_2^2[4])^{1/2} + \int_0^4 (u_1^2[t] + u_2^2[t])dt - \int_0^4 (\upsilon_1^2[t] + \upsilon_2^2[t])dt
\end{aligned} \tag{5.2}$$

After making the substitutions $x_1 = r_1, x_2 = \dot r_1, x_3 = r_2, x_4 = \dot r_2$, system (5.1) can be rewritten in the form (1.1), where we have to substitute

$$x = \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix}, \quad A = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -2 & -0.4 & 0.02 & 0 \\ 0 & 0 & 0 & 1 \\ 0.01 & 0 & -1 & -0.1 \end{Vmatrix}, \quad A_h = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ -1 & -0.4 & 0.4 & -1 \\ 0 & 0 & 0 & 0 \\ -0.3 & 0.7 & -0.4 & 0.5 \end{Vmatrix}$$

$$B(t)=\begin{Vmatrix} 0 & 0 \\ 5-t & 0 \\ 0 & 0 \\ 0 & 4-0.5t \end{Vmatrix}, \quad C(t)=\begin{Vmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & 3 \end{Vmatrix}, \quad u=\begin{Vmatrix} u_1 \\ u_2 \end{Vmatrix}, \quad v=\begin{Vmatrix} v_1 \\ v_2 \end{Vmatrix}$$

$$h=1, \quad t_0=0, \quad \vartheta=4$$

Here, the performance index (5.2) is rewritten in the form (1.2) in the case of the substitution

$$N=8, \quad t^{(i)}=0{,}5i, \quad g^{(i)}=0, \quad i=1,\dots,8, \quad D^{(1)}=D^{(8)}=E^{[4]}$$

$$D^{(2)}=\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}, \quad D^{(3)}=\begin{Vmatrix} 1 & 0 & 0 & 0 \end{Vmatrix}, \quad D^{(4)}=\begin{Vmatrix} 0 & 1 & 0 & 0 \end{Vmatrix}$$

$$D^{(5)}=\begin{Vmatrix} 0 & 0 & 0 & 1 \end{Vmatrix}, \quad D^{(6)}=\begin{Vmatrix} 0 & 0 & 1 & 0 \end{Vmatrix}, \quad D^{(7)}=\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$\Phi(t)=\Psi(t)=E^{[2]}$$

The problem of game control for system (5.1) in the case of performance index (5.2) was solved using the constructions described above. In this case, the dimensionality of the information object $y(\cdot)$ and, consequently, also of the auxiliary control problem of the w-system, $p = 16$.

We now present the results of modelling of the control process (5.1), (5.2) using a digital computer (in the calculations in scheme (1.3)–(1.5) we chose $k = 4000$, $\delta_k = 4.4000 = 0.001$ and $\varepsilon = 0.01$). The *a priori* calculated value of the game

$$\rho^0(x_0[-1[\cdot]0],\ u[0[\cdot]0),\ v[0[\cdot]0))=\rho^0(\{\sin\tau,\cos\tau,\cos\tau,-\sin\tau,-1\le\tau\le 0\})=\rho^0\approx 2.741$$

The motion of system (5.1), which occurred under the combined action of the optimal strategies $u^0(\cdot) = u^0(x[-1[\cdot]t], \varepsilon)$ and $\upsilon^0(\cdot) = \upsilon^0(x[-1[\cdot]t], \varepsilon)$ is shown in Fig. 1. In this case, the value of the performance index (5.2) is equal to

$$\gamma\approx 2.695+5.376-5.329=2.742\approx\rho^0$$

Graphs of the corresponding forms of the control and the disturbance are shown in Fig. 2.

The motion of system (5.1) which was realized under the action of the optimal control strategy $u^0(\cdot)$ in a pair with the disturbance $\upsilon(\cdot) = 1/2\Psi^{-1}(t)C^T(t)x(t)/|x(t)|$ is shown in Fig. 3. The result obtained is

$$\gamma\approx 2.281+4.278-4.033=2.526<\rho^0$$

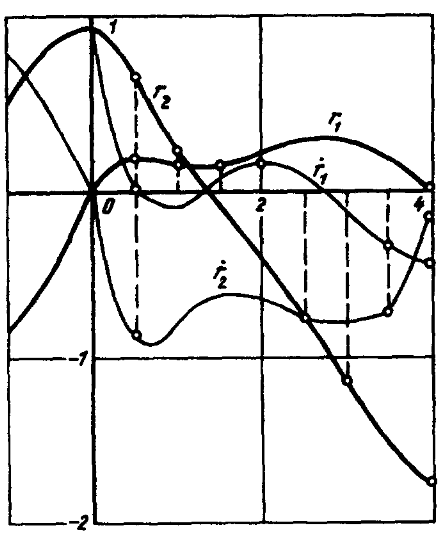

Fig. 1.

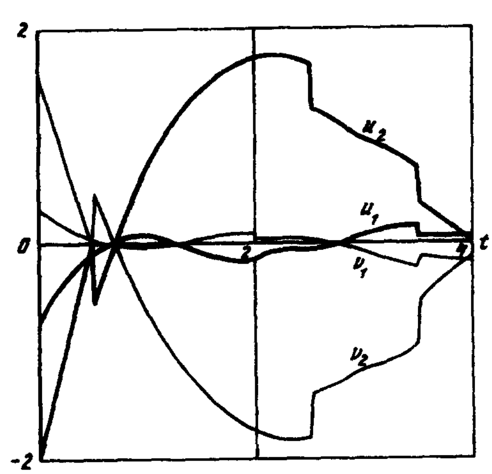

Fig. 2.

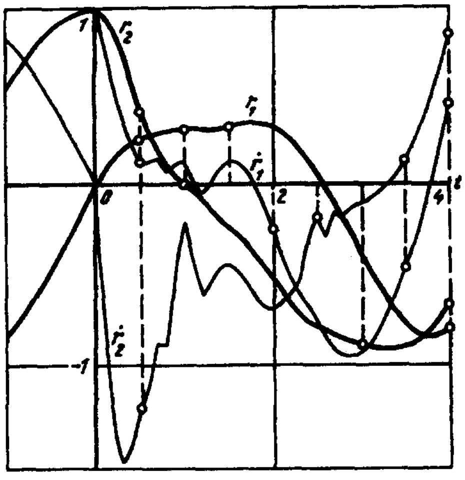

Fig. 3.

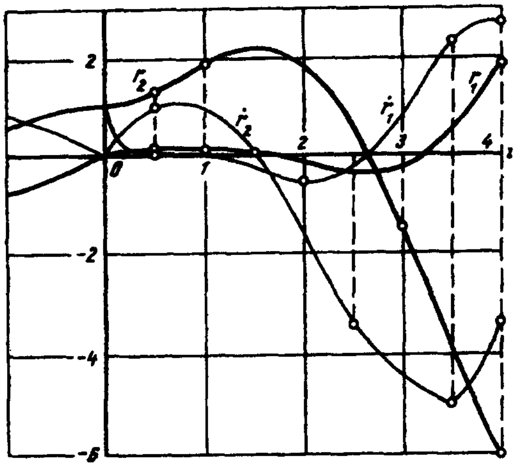

Fig. 4.

The motion of system (5.1), which occurred under the action of the optimal disturbance strategy $\upsilon^0(\cdot)$ in a pair with the control $u(\cdot) = -1/2\Phi^{-1}(t)B^T(t)x(t)/|x(t)|$ is shown in Fig. 4. The results obtained is

$$\gamma \approx 10.443 + 3.887 - 8.866 = 5.464 > \rho^0$$

The efficiency of the proposed control was also verified in the case of the following modelling of the control process (5.1), (5.2) as follows.

The optimal control strategy in a pair with the disturbance $\upsilon(\cdot) \equiv 0$, where the result was $\gamma \approx 1.195 + 1.502 - 0 = 2.697$.

The optimal control strategy $u^0(\cdot)$ in a pair with the disturbance $\upsilon(\cdot) = \{\upsilon_1(\cdot), \upsilon_2(\cdot)\} = \{0.6\sin(t), 0.4\sin(2t+1)\}$, where the result was $\gamma \approx 1.617 + 1.737 - 0.984 = 2.37$.

The optimal disturbance strategy $\upsilon^0(\cdot)$ in a pair with the control $u^0(\cdot) \equiv 0$, where the result was $\gamma \approx 32.344 + 0 - 12.166 = 20.178$.

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